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ABSTRACT

Combinatorial extension and composition methods have been extensively used in the construction of block designs. One of the composition methods, namely the direct product or Kronecker product method was utilized by Chakravarti [1956] to produce certain types of fractional factorial designs. The present paper shows how the direct sum operation can be utilized in obtaining from initial fractional factorial designs for two seperate symmetrical factorials a fractional factorial design for the corresponding asymmetrical factorial. Specifically, we provide some results which are useful in the construction of non-singular fractional factorial designs via the direct sum composition method. In addition a modified direct sum method is discussed and the consequences of imposing orthogonality are explored.

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FACTORIAL DESIGNS[1]

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1. INTRODUCTION

In design theory there are well known algebraic operations which lead to new designs, when we start out from a set of initial designs. For example, the direct product (or Kronecker product) operation was used by Chakravarti [1956] to produce fractional factorial designs for the asymmetrical factorial. The designs developed by him through this method did not relate to arbitrary initial factorial designs. These initial deisgns specifically arose from the existence of orthogonal arrays, which were much earlier shown to be quite useful in factorial design theory by Rao [1947].

To illustrate the direct product method we reproduce the following example, which follows immediately from theorem 1 of Chakravarti's [1956] paper. The orthogonal arrays D_1^* and D_2^* below are orthogonal main effect plans in $N_1=4$ and

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$N_2=9$ treatment combinations for the 2^3 and 3^4 factorials respectively.

	D_1^*	D_2^*		
	000	0000	1120	2210
(1.1)	110	0112	1202	
	101	0221	2022	
	011	1011	2101	

The direct product design $D_1^* \otimes D_2^*$ in $N_1N_2=36$ treatment combinations provides orthogonal estimates of not only the main effects but also of the two factor interaction of one 2-level factor with one 3-level factor for the $2^3 \times 3^4$ factorial under the assumption that all other effects are negligible.

$D_1^* \otimes D_2^*$

	0000000	0000112	0000221		0002210
(1.2)	1100000	1100112	1100221	...	1102210
	1010000	1010112	1010221		1012210
	0110000	0110112	0110221		0112210

In addition to the above type of designs there is a need to spell out the details of the direct product method for arbitrary initial desings and given arbitrary parameters under various assumptions on the total parametric vector. Such initial designs would encompass resolution III, IV and V designs.

In some settings (especially when the orthogonality condition is dropped) the resultant direct product design might be uneconomical from the viewpoint of number of treatment combinations. Thus, in the previous example, if main effects are the only ones of interest under the assumption that all interactions are negligible, it is clear that 36 treatment combinations are too many for estimation purposes. This is so because for the $2^3 \times 3^4$ factorial we need only 12 treatment combinations to form a main effect plan if no estimate of the error variance is desired. If it is desirable to have an estimate of the error variance as well then one or more treatment combinations can be added to the plan or one or more treatment combinations can be repeated in the plan.

To obtain economical fractions we can resort to a different operation altogether, e.g. we can compose two initial designs using the direct sum operation. Before taking a formal approach consider the main effect designs D_1 and D_2 consisting of $N_1=4$ and $N_2=6$ treatment combinations for the $k_1^{m_1}=2^2$ and $k_2^{m_2}=3^2$ factorials respectively:

(1.3)

D_1	D_2
10	00
01	22
11	10
11	12
	01
	21

It is easily verified, that the design $D_1 \oplus D_2$ below is a non-singular main effect plan in $N_1+N_2=4+6=10$ treatment combinations for the $k_1^{m_1} \times k_2^{m_2} = 2^2 \times 3^2$ asymmetrical factorial.

$$(1.4) \qquad \begin{array}{c|c} \multicolumn{2}{c}{D_1 \oplus D_2} \\ \hline 10 & \\ 01 & \\ 11 & A_2 \\ 11 & \\ \hline & 00 \\ & 22 \\ & 10 \\ A_1 & 12 \\ & 01 \\ & 21 \end{array} , \quad \text{where } A_1 = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} , \quad \text{and } A_2 = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \end{bmatrix} .$$

The operation involved in producing this design is clearly a direct sum type of operation, which we will call compactly *direct sum composition*. It is clear that the crucial part in using this method is the specification of the matrices A_1 and A_2. The choice of these will depend on what kind of properties one wishes to impose on the resulting design, given certain properties on the initial designs.

In the following sections we explore this new method in more detail and show how in some settings it always produces a design for the asymmetrical $k_1^{m_1} \times k_2^{m_2}$ factorial given the initial designs for the $k_1^{m_1}$ and $k_2^{m_2}$ factorials.

2. PRELIMINARY DEFINITIONS AND NOTATIONS

To make this paper relatively self-contained we introduce some of the basic notions of fractional replication. Consider the symmetric $k_1 \times k_2 \times \ldots \times k_t$ factorial consisting of t factors the i-th one having k_i levels. Let T denote the set of $N = \prod_{i=1}^{t} k_i$ treatment combinations.

DEFINITION 2.1. A factorial arrangement Γ with parameters $k_1, k_2, \ldots, k_t$; m; n; $r_1, r_2, \ldots, r_N$ is defined to be a collection of $n = \sum_{j=1}^{N} r_j$ treatments of T such that the j-th treatment in T has multiplicity (= replication number) r_j, and m is the number of nonzero r_j's.

DEFINITION 2.2 A factorial arrangement is said to be complete if $r_j > 0$ for all j.

DEFINITION 2.4. A factorial arrangement is said to be a fractional factorial arrangement, or simply a fractional replicate, if some but not all $r_j > 0$.

Let θ denote the $N \times 1$ vector of single degree of freedom parameters (also referred to as factorial effects) and let y_h be an observation at treatment h. The model we associate with T and θ is:

(2.1) $$E[Y_T] = X_T\theta, \quad Cov[Y_T] = \sigma^2 I$$

where an element of the N×1 vector of observations Y_T is y_h such that:

(2.2) $$E[y_h] = \theta' f(h).$$

Here f is an N-vector of k real valued known functions on the h's in T. We assume that the design matrix X_T is in orthonormal form, i.e. $X_T'X_T = I_N$. A familiar model of type (2.1) is the orthogonal-polynomial model.

Corresponding to a factorial arrangement Γ the model for the n×1 observation vector Y_Γ as induced by (2.1) is equal to:

(2.3) $$E[Y_\Gamma] = X_\Gamma\theta$$

where X_Γ is an n×N matrix simply read off from X_T taking repetitions into account.

To make the notation and forthcoming discussion simpler, let $G = \overset{t}{\underset{i=1}{X}} G_i$, (X = Carthesian product), where $G_i = \{0,1,2,.,k_i-1\}$.

Identify the elements of G_i with the levels of the i-th factor, so that G is a representation of T. With this representation a treatment or treatment combination g is then an element of G. Hence equations (2.1) and (2.2) may be rewritten as $E[Y_G]=X_G\theta$ and $E[y_g]=\theta' f(g)$, where $X_G=X_T$ and $f(g)=f(h)$. Thus when dealing with a fractional replicate Γ one is dealing with a collection of g's from G. Further, let an element of θ be indicated by the symbol $A_1^{u_1}A_2^{u_2}\ldots A_t^{u_t}$, where $(u_1,u_2,\ldots,u_t)$ is an element of G. This means that we have brought both the set of treatment combinations T and the set of factorial effects θ into one to one correspondence with G.

From the experimenter's viewpoint the most realistic partioning of the vector θ is the following:

$$\theta' = (\theta_1' \vdots \theta_2' \vdots \theta_3'), \tag{2.4}$$

where θ_1 is an N_1-vector to be estimated, θ_2 is an N_2-vector not of interest for estimation and not assumed to be known, and θ_3 is an N_3-vector assumed to be negligible (i.e. equal to zero), $1 \le N_1 \le N$, $0 \le N_2 \le N-1$, $0 \le N_3 \le N-1$ with $N_1+N_2+N_3=N$. The above partioning then leads to the following four cases:

$$(2.5) \qquad \begin{array}{rl} \text{(i)} & N_1=N,\ N_2=N_3=0 \\ \text{(ii)} & N_2=0,\ N_3\neq 0 \\ \text{(iii)} & N_2\neq 0,\ N_3\neq 0 \\ \text{(iv)} & N_2\neq 0,\ N_3=0 \end{array}$$

It is well known that case (ii) gives rise to odd resolution problems and case (iii) to even resolution problems. Case (i) leads to BLU estimation problem of θ and to response surface fitting problems while cases (iii) and (iv) bring about the biased estimation problems.

3. THE SUM COMPOSITION METHOD

Consider two separate symmetric factorials, i.e. the $k_1\times k_1\times\ldots\times k_1=k_1^{m_1}$ and the $k_2\times k_2\times\ldots\times k_2=k_2^{m_2}$ factorials and consider for each factorial the partitioning in (2.4). Explicitly let the $k_2^{m_i}\times 1$ parametric vector β_i be partitioned as:

$$(3.1) \qquad \beta_i' = (\beta_{i1}' \vdots \beta_{i2}' \vdots \beta_{i3}'), \quad i = 1, 2.$$

where β_{i1} is the $p_{i1}\times 1$ vector of parameters to be estimated, β_{i2} is the $p_{i2}\times 1$ vector of parameters not of interest and not assumed to be zero, and, β_{i3} is the $(k_i^{m_i}-p_{i1}-p_{i2})\times 1$ vector of parameters assumed to be zero. We assume the first element of both β_{11} and β_{21} in respectively the $k_1^{m_1}$ and $k_2^{m_2}$

factorials to be equal to the mean μ. Also we limit ourselves in this section to the most popular case, i.e. the odd resolution case, where $p_{12}=p_{22}=0$. Note that this is case (ii) of the previous section for two separate symmetrical factorials.

Let D_i=1,2, be a design consisting of N_i treatment combinations from the $k_i^{m_i}$ factorial such that the vector β_i is estimable. Further let $\beta_{11}\cup\beta_{21}$ be the $(p_{11}+p_{21}-1)\times 1$ parametric vector whose entries are elements of the union of β_{11} and β_{21}, when these are considered as sets of parameters in the $k_1^{m_1}\times k_2^{m_2}$ mixed factorial setup. Consider the design:

$$(3.1) \qquad D_1 \oplus D_2 = \begin{array}{c:c} D_1 & A_2 \\ \hdashline A_1 & D_2 \end{array},$$

where A_1 is $N_2\times m_1$ and A_2 is $N_1\times m_2$, and the rows of A_i are treatment combinations from the $k_i^{m_i}$ factorial. We desire a choice of A_1 and A_2 such that the resulting design provides unbiased estimates for the elements of $\beta_{11}\cup\beta_{21}$.

DEFINITION 3.1. Given designs D_1 and D_2 such that β_{11} and β_{21} are estimable in the symmetrical factorials $k_1^{m_1}$ and $k_2^{m_2}$ respectively, then a design $D_1\oplus D_2$ consisting of N_1+N_2 treatment combinations such that $\beta_{11}\cup\beta_{21}$ is estimable in the $k_1^{m_1}\times k_2^{m_2}$ mixed factorial, is said to be obtained using the direct sum composition of design D_1 and D_2.

It is clear that the crucial part in coming up with a design $D_1 \oplus D_2$ such that $\beta_{11} \cup \beta_{21}$ is estimable is the specification of the matrices A_1 and A_2, because they will have to guarantee the nonsingularity of the underlying design matrix of $D_1 \oplus D_2$. In other words, the selection of A_1 and A_2 should be such that rank of the design matrix of $D_1 \oplus D_2$ is equal to $p_{11}+p_{21}-1$. Before providing some nonsingular constructions of direct sum designs, we first proceed to prove an algebraic theorem which is of a fundamental nature.

Let $D_1 \oplus D_2$ be a direct sum design, then its design matrix $X_{D_1 \oplus D_2}$ has the following structure by virtue of D_1, D_2 and $\beta_{11} \cup \beta_{21}$ in the $k_1^{m_1} \times k_2^{m_2}$ mixed factorial setup:

$$
(3.2) \qquad X_{D_1 \oplus D_2} = \left[\begin{array}{c|cccc|cccc}
c & e_{11} & e_{12} & \cdots & e_{1p_{11}-1} & f_{11} & f_{12} & \cdots & f_{1p_{21}-1} \\
c & e_{21} & e_{22} & \cdots & e_{2p_{11}-1} & f_{21} & f_{22} & \cdots & f_{2p_{21}-1} \\
\vdots & & & \vdots & & & & \vdots & \\
c & e_{N_1 1} & e_{N_1 2} & \cdots & e_{N_1 p_{11}-1} & f_{N_1 1} & f_{N_1 2} & \cdots & f_{N_1 p_{21}-1} \\
\hline
c & g_{11} & g_{12} & \cdots & g_{1p_{11}-1} & h_{11} & h_{12} & \cdots & h_{1p_{21}-1} \\
c & g_{21} & g_{22} & \cdots & g_{2p_{11}-1} & h_{21} & h_{22} & \cdots & h_{2p_{21}-1} \\
\vdots & & & \vdots & & & & \vdots & \\
c & g_{N_2 1} & g_{N_2 2} & \cdots & g_{N_2 p_{11}-1} & h_{N_2 1} & h_{N_2 2} & \cdots & h_{N_2 p_{21}-1}
\end{array}\right]
$$

$$(3.3) \qquad = \left[\begin{array}{c|c|c} c\mathbf{1}_{N_1} & E & F \\ \hline c\mathbf{1}_{N_2} & G & H \end{array}\right].$$

Note that the matrices E and H are essentially determined by D_1 and D_2 and $\beta_{11}\cup\beta_{21}$ in the context of the $k_1^{m_1}\times k_2^{m_2}$ mixed factorial. Let us now form the following vectors of order N_1+N_2:

$$(3.4) \qquad \begin{cases} v_0' = (c,c,\ldots,c,0,0,\ldots,0) \\ w_0' = (0,0,\ldots,0,c,c,\ldots,c) \\ v_i' = (e_{1i},e_{2i},\ldots,e_{N_1 i},0,0,\ldots,0), \ i=1,2,\ldots,p_{11}-1 \\ w_i' = (0,0,\ldots,0,g_{1i},g_{2i},\ldots,g_{N_2 i}), \ i=1,2,\ldots,p_{11}-1 \\ y_j' = (f_{1j},f_{2j},\ldots,f_{Nj},0,0,\ldots,0), \ j=1,2,\ldots,p_{21}-1 \\ z_j' = (0,0,\ldots,0,h_{1j},h_{2j},\ldots,h_{N_2 j}), \ j=1,2,\ldots,p_{21}-1. \end{cases}$$

Denote the columns of $X_{D_1\oplus D_2}$ by $x_0,x_1,x_2,\ldots,x_{p_{11}-1},x_1^*,x_2^*,\ldots,x_{p_{21}-1}^*$, then it is seen that:

$$(3.5) \qquad \begin{cases} x_0 = v_0' + w_0' \\ x_i' = v_i' + w_i', \ i=1,2,\ldots,p_{11}-1 \\ x_j^{*\prime} = y_j' + z_j', \ j=1,2,\ldots,p_{21}-1 \end{cases}$$

Let χ be the subspace spanned by the vectors $x_0, x_1, x_2, \ldots, x_{p_{11}-1}$ and χ^* the subspace spanned by $x_1^*, x_2^*, \ldots, x_{p_{21}-1}^*$, then the dimension of the space $\chi+\chi^*$ spanned by the two sets of vectors is equal to:

$$\dim(\chi+\chi^*) = \dim\chi+\dim\chi^*-\dim\chi\cap\chi^*. \tag{3.6}$$

Now, $X_{D_1 \oplus D_2}$ will have full column rank (i.e. rank of $X_{D_1 \oplus D_2} = p_{11}+p_{21}-1$) if and only if $\dim\chi\cap\chi^*$ is equal to zero. On the other hand for a vector x to be in $\chi\cap\chi^*$, it must be a linear combination of the set of vectors $x_0, x_1, x_2, \ldots, x_{p_{11}-1}$ and it must also be a linear combination of the set of vectors x_1^*, $x_2^*, \ldots, x_{p_{21}-1}^*$. This means that:

$$\begin{cases} x = \alpha_0 x_0+\alpha_1 x_1+\alpha_2 x_2+\ldots+\alpha_{p_{11}-1} x_{p_{11}-1} \\ \quad = \lambda_1 x_1^*+\lambda_2 x_2^*+\lambda_3 x_3^*+\ldots+\lambda_{p_{21}-1} x_{p_{21}-1}^* \end{cases} . \tag{3.7}$$

Utilizing (3.5) we see that (3.7) implies:

$$\begin{cases} \alpha_0(v_0+w_0)+\alpha_1(v_1+w_1)+\alpha_2(v_2+w_2)+\ldots+\alpha_{p_{11}-1}(v_{p_{11}-1}+w_{p_{11}-1}) \\ =\lambda_1(y_1+z_1)+\lambda_2(y_2+z_2)+\ldots+\lambda_{p_{21}-1}(y_{p_{21}-1}+z_{p_{21}-1}). \end{cases} \tag{3.8}$$

From the definition of the vectors v_0', w_0', v_i', w_i', y_j' and z_j' it follows that:

$$(3.9)\quad \begin{cases} \alpha_0 v_0+\alpha_1 v_1+\alpha_2 v_2+\ldots+\alpha_{p_{11}-1} v_{p_{11}-1}=\lambda_1 y_1+\lambda_2 y_2+\ldots+\lambda_{p_{21}-1} y_{p_{21}-1} \\ \alpha_0 w_0+\alpha_1 w_1+\alpha_2 w_2+\ldots+\alpha_{p_{11}-1} w_{p_{11}-1}=\lambda_1 z_1+\lambda_2 z_2+\ldots+\lambda_{p_{21}-1} z_{p_{21}-1} \end{cases}$$

Hence for $\dim(\chi+\chi^*)$ to be less than $p_{11}+p_{11}-1$, the two equations in (3.9) must be satisfied non-trivially.

The above development immediately leads us to:

THEOREM 3.1. *The design matrix* $X_{D_1 \oplus D_2}$ *of the direct sum design* $D_1 \oplus D_2$ *is of full column rank if and only if the two equations in (3.9) are not satisfied non-trivially.*

The meaning of this theorem is that if the matrices A_1 and A_2 in (3.1) are chosen such that the vectors in (3.4) do not satisfy the equations in (3.9) then the resulting direct sum will be capable of providing unbiased estimates of $\beta_{11} \cup \beta_{21}$. A sufficient condition for a nonsingular construction is provided in the following corrollary.

COROLLARY 3.1. *If* D_1 *and* D_2 *are designs such that* β_{11} *and* β_{21} *are estimable in the separate* $k_1^{m_1}$ *and* $k_2^{m_2}$ *symmetrical factorials respectively, and* A_1 *and/or* A_2 *are chosen to consist of repetitions of a fixed treatment combination then the direct sum design* $D_1 \oplus D_2$ *is such that* $\beta_{11} \cup \beta_{21}$ *is estimable in the mixed* $k_1^{m_1} \times k_2^{m_2}$ *factorial.*

PROOF: Suppose that A_1 consists of repetitions of a fixed treatment combination from the $k_1^{m_1}$ symmetrical factorial and that A_2 consists of arbitrary treatment combinations from the $k_2^{m_2}$ factorial. Then from (3.2) and (3.3) it follows that each column of G is a constant vector, i.e. a vector whose entries are equal to a scalar. This implies that in (3.9) $\alpha_0 w_0+\alpha_1 w_1+\alpha_2 w_2+\ldots+\alpha_{p_{11}-1} w_{p_{11}-1}$ is a constant vector, which in turn implies that $z_1, z_2, \ldots, z_{p_{21}-1}$ are linearly dependent. This contradicts the hypothesis of independence of the z's guaranteed by D_2. Hence the second equation in (3.9) cannot be satisfied, so that by Theorem 3.1. $X_{D_1 \oplus D_2}$ has full column rank and therefore $\beta_{11} \cup \beta_{21}$ is estimable.

A similar argument will show that the first equation in (3.9) cannot be satisfied if A_2 consists of repetitions of a fixed treatment combination from the $k_2^{m_2}$ symmetrical factorial. Finally, if both A_1 and A_2 consist of repetitions of a fixed treatment combination from their respective factorials, then both equations of (3.9) cannot be satisfied and hence $X_{D_1 \oplus D_2}$ will be of full rank. This completes the proof.

REMARK 3.1. The question arises, whether one may start with a D_1 or D_2 in (3.1) such that the design matrices E or H are less than full column rank and by suitable choices of A_1 and A_2 still come up with a direct sum design $D_1 \oplus D_2$ such that $X_{D_1 \oplus D_2}$ is of full column rank. That this is possible is illustrated in the following example.

Consider the 2^2 and 3^2 factorials and suppose that we are interested in main effects only under the orthogonal polynomial set up. Consider the direct sum design:

$$(3.10) \qquad D_1 \oplus D_2 = \left[\begin{array}{c|c} D_1 & A_2 \\ \hline A_1 & D_2 \end{array}\right] = \left[\begin{array}{c|c} 00 & 10 \\ 01 & 01 \\ 10 & 11 \\ \hline 11 & 00 \\ 11 & 10 \\ 11 & 20 \\ 11 & 12 \\ 11 & 22 \end{array}\right]$$

Note that the design D_2 gives to rise a design matrix which has rank less than $P_{21} = 2(2)+1 = 5$, since in the 3^2 orthogonal polynomial set up the design matrix

$$X_D = \begin{bmatrix} \frac{1}{3} & -\frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & -\frac{2}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{3} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & -\frac{2}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{3} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} \end{bmatrix} \tag{3.11}$$

clearly has linearly dependent columns. However, the design matrix $X_{D_1 \oplus D_2}$ of the direct sum design $D_1 \oplus D_2$ is of full column rank, namely $p_{11}+p_{21}-1=3+5-1=7$, since

$$X_{D_1 \oplus D_2} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 0 & -\frac{1}{3\sqrt{2}} & -\frac{1}{2\sqrt{6}} & \frac{1}{6\sqrt{2}} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & -\frac{1}{2\sqrt{6}} & \frac{1}{6\sqrt{2}} & 0 & -\frac{1}{3\sqrt{2}} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & 0 & -\frac{1}{3\sqrt{2}} & 0 & -\frac{1}{3\sqrt{2}} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{2\sqrt{6}} & \frac{1}{6\sqrt{2}} & -\frac{1}{2\sqrt{6}} & \frac{1}{6\sqrt{2}} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & -\frac{1}{3\sqrt{2}} & -\frac{1}{2\sqrt{6}} & \frac{1}{6\sqrt{2}} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2\sqrt{6}} & \frac{1}{6\sqrt{2}} & -\frac{1}{2\sqrt{6}} & \frac{1}{6\sqrt{2}} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & -\frac{1}{3\sqrt{2}} & \frac{1}{2\sqrt{6}} & \frac{1}{6\sqrt{2}} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2\sqrt{6}} & \frac{1}{6\sqrt{2}} & \frac{1}{2\sqrt{6}} & \frac{1}{6\sqrt{2}} \end{bmatrix} \tag{3.12}$$

has seven linearly independent columns.

4. CLASSES OF SIMILAR DESIGNS

In this section we show that designs obtained via the sum composition method can be used to produce classes of *similar* designs if the parametric vector $\beta_{11} \cup \beta_{21}$ satisfies certain regularity conditions, which are typically met in many practical applications.

Let us now apply some of the ideas laid down in two recent papers by Srivastava, Raktoe and Pesotan [1970] and by Pesotan, Raktoe and Federer [1972].

In section 2, we introduced the set G as the set of treatment combinations of order $\prod_{i=1}^{t} k_i$. In the present setting G has cardinality $(k_1^{m_1})(k_2^{m_2})$, since we are dealing with direct sum designs from two symmetrical factorials. We denote an element of G by the (m_1+m_2)-tuple $(x_{11},x_{12},\ldots,x_{1m_1},x_{21},x_{22},\ldots,x_{2m_2})$, where x_{ij} is an element of $G_i=\{0,1,2,\ldots,k_i-1\}$, $i=1,2, j=1,2,\ldots,m_i$. Further, let an element of $\theta=\beta_1 \cup \beta_2$ be denoted by the symbol $A_{11}^{x_{11}}A_{12}^{x_{12}}\ldots A_{1m_1}^{x_{1m_1}}A_{21}^{x_{21}}A_{22}^{x_{22}}\ldots A_{2m_2}^{x_{2m_2}}$. Note that in this notation the mean $\mu=A_{11}^0A_{12}^0\ldots A_{m_1}^0A_{21}^0A_{22}^0\ldots A_{2m_2}^0$ and $\{A_{11}^1A_{12}^0\ldots A_{1m_1}^0A_{21}^0A_{22}^0\ldots A_{2m_2}^0, A_{11}^2A_{12}^0\ldots A_{1m_1}^0A_{21}^0A_{22}^0\ldots A_{2m_2}^0, \ldots A_{11}^{k_1-1}A_{12}^0\ldots A_{1m_1}^0A_{21}^0A_{22}^0\ldots A_{2m_2}^0\}$ represents the set of k_1-1 normalized orthogonal parametric contrasts associated with the first factor, etc.

DEFINITION 4.1. Two designs D_1 and D_2 from the $k_1^{m_1} \times k_2^{m_2}$ mixed factorial are said to be similar with respect to a sub-vector of θ if and only if the underlying infirmation matrices $X'_{D_1} X_{D_1}$ and $X'_{D_2} X_{D_2}$ have the same spectrum.

DEFINITION 4.2. A parametric sub vector θ^* of θ is said to be admissable if and only if whenever $A_{11}^{x_{11}} A_{12}^{x_{12}} \ldots A_{ij}^{x_{ij}} \ldots A_{2m_2}^{x_{2m_2}}$ belongs to θ^* **and** $x_{ij} \neq 0$, then $A_{11}^{x_{11}} A_{12}^{x_{12}} \ldots A_{ij}^{r} \ldots A_{2m_2}^{x_{2m_2}}$ belongs to θ^* for all $r \neq 0$.

Let Ω be the group of level permutations acting on the elements of G. An element ω of Ω is of the form

$$\omega = (\omega_1, \omega_2) = (\omega_{11}, \omega_{12}, \ldots, \omega_{1m_1}, \omega_{21}, \omega_{22}, \ldots, \omega_{2m_2}) \tag{4.1}$$

such that $\omega_i = (\omega_{i1}, \omega_{i2}, \ldots, \omega_{im_i})$ and ω_{ij} is a permutation acting on G_i, $i=1,2$, $j=1,2,\ldots,m_i$.

The following lemmas are special cases from Srivastava, Raktoe and Pesotan [1971].

LEMMA 4.1. *If* D *is an arbitrary design from the* $k_1^{m_1} \times k_2^{m_2}$ *mixed factorial and* $\omega(D)$ *is the permuted design obtained from* D *and* θ^* *is admissible, then there exists an orthogonal matrix* P_ω *such that*

$$X_{\omega(D)} = X_D P_\omega \tag{4.2}$$

where X_D *and* $X_{\omega(D)}$ *are design matrices of the fractions* D *and* $\omega(D)$ *with respect to* θ^*.

LEMMA 4.2. *If θ^* and θ^{**} are two admissable subvectors of θ, then the vector $\theta^* \cup \theta^{**}$ obtained by set theroetic union of θ^* and θ^{**} is also admissable.*

These two lemmas together with the definition of ω in (4.1) lead us immediately to the following theorem.

THEOREM 4.1. *Let D_1 and D_2 be two fractions from the $k_1^{m_1}$ and $k_2^{m_2}$ factorial respectively for the admissable vectors β_{11} and β_{21}. Let $\omega_1(D_1)$ and $\omega_2(D_2)$ be the fractions obtained from D_1 and D_2 be the actions of ω_1 and ω_2 respectively. If*

$$(4.3) \qquad D_1 \oplus D_2 = \begin{array}{c|c} D_1 & A_2 \\ A_1 & D_2 \end{array} \quad \text{and} \quad \omega_1(D_1) \oplus \omega_2(D_2) = \begin{array}{c|c} \omega_1(D_1) & \omega_2(A_2) \\ \omega_1(A_1) & \omega_2(D_2) \end{array}$$

are the two direct sum designs for $\beta_{11} \cup \beta_{21}$ obtained by sum composition and action of ω_1 and ω_2 respectively then $D_1 \oplus D_2$ and $\omega_1(D_1) \oplus \omega_2(D_2)$ are similar designs, i.e. the underlying information matrices $X'_{D_1 \oplus D_2} X_{D_1 \oplus D_2}$ and $X'_{\omega_1(D_1) \oplus \omega_2(D_2)} X_{\omega_1(D_1) \oplus \omega_2(D_2)}$ have the same characteristic roots.

PROOF: By lemma 4.2. we know that $\beta_{11} \cup \beta_{21}$ is admissable and from (4.1) we know that $\omega = (\omega_1, \omega_2)$, which implies that $\omega_1(D_1) \oplus \omega_2(D_2) = \omega(D_1 \oplus D_2)$. Hence by lemma 4.1 there exists an orthogonal matrix P_ω such that

(4.4) $$X_{\omega(D_1 \oplus D_2)} = X_{D_1 \oplus D_2} P_\omega .$$

Therefore the underlying information matrices of $D_1 \oplus D_2$ and $\omega(D_1 \oplus D_2)$ have the same spectrum.

The meaning of theorem 4.1 is that when a design is obtained using the sum composition method one has also obtained a class of designs each one providing the same amount of information, if this quantity is taken as a functional on the spectrum of the information matrix. In a recent paper Pesotan, Raktoe and Federer [1972] have obtained some characterization and enumeration results on classes of similar designs in a very general setting. These results can be applied in the present setting as well.

As an illustration of theorem 4.1 consider the direct sum design for main effects of the $2^2 \times 3^2$ factorial discussed in the introductory part of this paper. If ω is taken to be $\omega=(\omega_1,\omega_2)=(\omega_{11},\omega_{12},\omega_{21},\omega_{22})$ such that:

(4.5) $$\omega_{11}: \begin{matrix}0\\ \\1\end{matrix} \to \begin{matrix}0\\ \\1\end{matrix}, \quad \omega_{12}: \begin{matrix}0\\ \\1\end{matrix} \to \begin{matrix}1\\ \\0\end{matrix}, \quad \omega_{21}: \begin{matrix}0\\1\\2\end{matrix} \to \begin{matrix}1\\2\\0\end{matrix}, \quad \omega_{22}: \begin{matrix}0\\1\\2\end{matrix} \to \begin{matrix}0\\1\\2\end{matrix}$$

then the designs $D_1 \oplus D_2$ and $\omega_1(D_1) \oplus \omega_2(D_2)$ below:

$$(4.6)\qquad \begin{array}{c|c} \multicolumn{2}{c}{\underline{D_1 \oplus D_2}} \\ 10 & 20 \\ 01 & 20 \\ 11 & 20 \\ 11 & 20 \\ \hdashline 10 & 00 \\ 10 & 22 \\ 10 & 10 \\ 10 & 12 \\ 10 & 01 \\ 10 & 21 \end{array} \quad , \quad \begin{array}{c|c} \multicolumn{2}{c}{\underline{\omega_1(D_1) \oplus \omega_2(D_2)}} \\ 11 & 00 \\ 00 & 00 \\ 10 & 00 \\ 10 & 00 \\ \hdashline 11 & 10 \\ 11 & 02 \\ 11 & 20 \\ 11 & 22 \\ 11 & 11 \\ 11 & 01 \end{array}$$

are such that the underlying information matrices have the same spectrum.

5. A MODIFIED SUM COMPOSITION METHOD

The sum composition method as discussed in section 3 results always in a direct sum design which has one treatment combination too many. This comes from the fact that in the individual designs each time the mean is to be estimated. In the saturated case the number of treatment combinations should equal the number of parameters to be estimated in the mixed factorial, i.e. N_1+N_2 should be equal to $p_{11}+p_{21}-1$. The direct sum procedure of section 3 would in this case provide $p_{11}+p_{12}$ treatment combinations, i.e. one two many.

Consider the design D_i such that p_{i1} parameters are estimable in the $k_i^{m_i}$ factorial. Suppose that β_{i1} is admissable and that D_i contains the treatment combination $(x_{i1}^*, x_{i2}^*, \ldots, x_{im_i}^*)$, i=1,2. From the development in the previous chapter we know that a similar design $\omega_i^*(D_i)$ can be obtained from D_i such that when the permutation ω_i^* is applied to each of the treatment-combinations in D_i it takes $(x_{i1}^*, x_{i2}^*, \ldots, x_{im_i}^*)$ into $(0,0,\ldots,0)$. Hence $\omega_1^*(D_1)$ and $\omega_2^*(D_2)$ can be written as:

$$\text{(5.1)} \qquad \begin{array}{c} \omega_1^*(D_1) \\ \hline 00\ldots0 \\ D_1^* \end{array} \qquad \begin{array}{c} \omega_2^*(D_2) \\ \hline 00\ldots0 \\ D_2^* \end{array}$$

where D_i^* consists of the remainder of the permuted treatment combinations of $\omega_i^*(D_i)$. Following Definition 3.1 we may now define the direct sum of designs D_1 and D_2 as:

$$\text{(5.2)} \qquad D_1 \oplus D_2 = \omega_1^*(D_1) \oplus \omega_2^*(D_2) = \begin{array}{c:c} D_1^* & A_2 \\ \hdashline 00\ldots0 & 00\ldots0 \\ \hdashline A_1 & D_2^* \end{array}$$

where A_i consists of N_i-1 treatment combinations from the $k_i^{m_i}$ factorial. Note, that such a direct sum design has $N_1-1+N_2-1+1=N_1+N_2-1$ treatment combinations, yielding a saving of one treatment combination over the design in (3.1).

As an illustration, consider the designs introduced in the introduction and further discussed in section 4. Taking the permutations $\omega_1^*=(\omega_{11}^*,\omega_{12}^*)$, $\omega_2^*=(\omega_{21}^*,\omega_{22}^*)$ as:

$$(5.3)\qquad \omega_{11}^*: \begin{matrix}0\\ 1\end{matrix} \to \begin{matrix}1\\ 0\end{matrix}\ ,\ \omega_{12}^*: \begin{matrix}0\\ 1\end{matrix} \to \begin{matrix}1\\ 0\end{matrix}\ ,\ \omega_{21}^*: \begin{matrix}0\\ 1\\ 2\end{matrix} \to \begin{matrix}0\\ 1\\ 2\end{matrix}\ ,\ \omega_{22}^*: \begin{matrix}0\\ 1\\ 2\end{matrix} \to \begin{matrix}0\\ 1\\ 2\end{matrix}$$

we obtain:

$$(5.4)\qquad \begin{array}{c}\omega_1^*(D_1)\\ \hline 00\\ \cdots\cdots \\ D_1^*=\left\{\begin{array}{l}01\\ 10\\ 00\end{array}\right. \\ \\ \\ \\ \end{array} \qquad \text{and} \qquad \begin{array}{c}\omega_2^*(D_2)\\ \hline 00\\ \cdots\cdots \\ D_2^*=\left\{\begin{array}{l}22\\ 10\\ 12\\ 01\\ 21\end{array}\right.\end{array}$$

Utilizing (5.2) we then may produce a direct sum design in the following way:

$$(5.5)\qquad D_1 \oplus D_2=\omega_1^*(D_1)+\omega_2^*(D_2)= \begin{array}{c:c} 01 & \\ 10 & A_2 \\ 00 & \\ \hdashline 00 & 00 \\ \hdashline & 22 \\ A_1 & 10 \\ & 12 \\ & 01 \\ & 21 \end{array}$$

where A_1 consists of N_1-1=6-1=5 treatment combinations from the 2^2 factorial and A_2 consists of N_2-1=4-1=3 treatment combinations from the 3^2 factorial.

As a further example, consider the main effect plans:

$$(5.6) \qquad D_1 = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}, \text{ and } D_2 = \begin{array}{c} 1000 \\ 0111 \\ 1011 \\ 0001 \\ 0010 \end{array}$$

from the 4^1 and 2^4 symmetrical factorials respectively. Let

$$(5.7) \qquad \omega_1^*: \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \rightarrow \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \text{ and } \omega_{21}^*: \begin{array}{c} 0 \\ 1 \end{array} \rightarrow \begin{array}{c} 1 \\ 0 \end{array}, \ \omega_{22}^*: \begin{array}{c} 0 \\ 1 \end{array} \rightarrow \begin{array}{c} 0 \\ 1 \end{array}, \ \omega_{23}^*: \begin{array}{c} 0 \\ 1 \end{array} \rightarrow \begin{array}{c} 0 \\ 1 \end{array}, \ \omega_{24}^*: \begin{array}{c} 0 \\ 1 \end{array} \rightarrow \begin{array}{c} 0 \\ 1 \end{array}.$$

then

$$(5.8) \qquad D_1 \oplus D_2 = \omega_1^*(D_1) \oplus \omega_2^*(D_2) = \begin{array}{c|c} 1 & \\ 2 & A_2 \\ 3 & \\ \hline 0 & 0000 \\ \hline & 1111 \\ & 0011 \\ A & 1001 \\ & 1010 \end{array}$$

is a direct sum design consisting of 8 treatment combinations from the 4×2^4 mixed factorial if A_1 and A_2 are chosen properly. For example, if

$$(5.9) \qquad A_1 = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}, \quad \text{and} \quad A_2 = \begin{array}{c} 0101 \\ 0110 \\ 1100 \end{array}$$

then the resulting direct sum design provides an orthogonal main effect plan in the sense of Addelman [1972], whose design on page 104 suggested the selected A_1 and A_2.

The algebraic and construction aspects of the direct sum design in (5.2) can be explored along the same lines as in Theorem 3.1 and Corollary 3.1.

6. SOME DESIRABLE PROPERTIES

Sofar in the development we have ignored the various properties, which come typically into play when dealing with fractional replication. The first one is the property of *orthogonality*. Suppose that this property is imposed on the direct sum design $D_1 \oplus D_2$ in (3.1). This means that

(6.1) $$X'_{D_1\oplus D_2} X_{D_1\oplus D_2} = \text{diagonal matrix} = M, \text{ say},$$

so that from (3.3) we must have:

(6.2) $$\left[\begin{array}{c|c} c\mathbf{1}'_{N_1} & c\mathbf{1}'_{N_2} \\ \hline E' & G' \\ \hline F' & H' \end{array}\right] \left[\begin{array}{c|c|c} c\mathbf{1}_{N_1} & E & F \\ \hline c\mathbf{1}_{N_2} & G & H \end{array}\right] = \left[\begin{array}{c|c|c} N_1+N_2 & c\mathbf{1}'_{N_1}E+c\mathbf{1}'_{N_2}G & c\mathbf{1}'_{N_1}F+c\mathbf{1}'_{N_2}H \\ \hline cE\mathbf{1}_{N_1}+cG'\mathbf{1}_{N_2} & E'E+G'G & E'F+G'H \\ \hline cF'\mathbf{1}_{N_1}+cH'\mathbf{1}_{N_2} & F'E+H'G & F'F+H'H \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} c^* & 0 & 0 \\ \hline 0 & M_1 & 0 \\ \hline 0 & 0 & M_2 \end{array}\right]$$

where $c^{*}=N_1+N_2$ and the M_i's are diagonal matrices of appropriate dimensions comprising M. Equation (6.2) implies the following conditions:

$$
(6.3) \qquad \begin{cases} E'E+G'G=M_1 \\ F'F+H'H=M_2 \\ \mathbf{1}'_{N_1}E+\mathbf{1}'_{N_2}G=0 \\ \mathbf{1}'_{N_1}F+\mathbf{1}'_{N_2}H=0 \\ E'F+G'H=0 \end{cases}
$$

By suitable selections of designs D_1, D_2, A_1 and A_2 in (3.1) one may satisfy the conditions in (6.3) and obtain an orthogonal direct sum design. It is clear that the conditions in (6.3) are in addition to the condition that the equations in (3.9) are not satisfied.

Along these same lines conditions can be formulated such that the direct sum design $D_1 \oplus D_2$ is *variance balanced* or *bias balanced* [see Hedayat, Raktoe and Federer [1972]). These properties along with the orthogonality property becomes especially meaningful, if we are dealing with special designs, such as the resolution III and V designs.

7. DISCUSSION

As is apparent from section 6 we just have initiated thoughts on the direct sum method. Considerable work is necessary to exploit in full the ramifications of this method, especially in the odd and even resolution settings.

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